Equivalence of the 8-vertex model on a Kagome lattice with the 32-vertex model on a triangular lattice

1976 J. Phys. A: Math. Gen. 9 L183

(http://iopscience.iop.org/0305-4470/9/12/003)

J. Phys. A: Math. Gen., Vol. 9, No. 12, 1976. Printed in Great Britain. © 1976

LETTER TO THE EDITOR

Equivalence of the 8-vertex model on a Kagomé lattice with the 32-vertex model on a triangular lattice

K Y Lin†

Physics Department, National Tsing Hua University, Hsinchu, Taiwan, Republic of China

Received 27 July 1976

Abstract. It is shown that the 8-vertex model on a Kagomé lattice is equivalent to the 32-vertex model on a triangular lattice. In particular, the ice-rule vertex model on a Kagomé lattice is equivalent to the 20-vertex model on a triangular lattice.

The 8-vertex model on a square lattice was solved by Baxter (1971). The 8-vertex model on a Kagomé lattice was considered by Lin (1976), and the soluble case of a free-fermion model was solved by the Pfaffian method. The Pfaffian solution of the 32-vertex model on a triangular lattice was given by Sacco and Wu (1975). The F model on a triangular lattice was considered by Baxter (1969) and can be solved exactly when the vertex weights satisfy a special condition (which includes the triangular ice model).

We shall demonstrate that the 8-vertex model on a Kagomé lattice is equivalent to the 32-vertex model on a triangular lattice. In particular, the ice-rule vertex model on a Kagomé lattice (Lin 1975, Lin and Tang 1976) is equivalent to the ice-type 20-vertex model on a triangular lattice (Kelland 1974a, b). The Rys F model (Rys 1963) is formulated on a Kagomé lattice. For certain values of the vertex weights the Kagomé lattice F model is equivalent to the soluble F model on a triangular lattice.

Place arrows on the bonds of a Kagomé lattice and allow only those configurations with an even number of arrows pointing into each vertex. The three sublattices are denoted by A, B and C, as shown in figure 1. The eight possible configurations allowed at each vertex and the corresponding vertex weights are shown in figure 2. Let the vertex weights be

$$\begin{aligned}\{\omega\}&=\{\omega_1,\omega_2,\ldots,\omega_8\} &&\text{on A}\\ \{\omega'\}&=\{\omega'_1,\omega'_2,\ldots,\omega'_8\} &&\text{on B}\\ \{\omega''\}&=\{\omega''_1,\omega''_2,\ldots,\omega''_8\} &&\text{on C.}\end{aligned} \tag{1}$$

The partition function is

$$Z=\sum\Big(\prod\omega_i^{n_i}\Big)\Big(\prod\omega_i'^{n'_i}\Big)\Big(\prod\omega_i''^{n''_i}\Big) \tag{2}$$

where the summation is extended to all allowed arrow configurations, and $n_i(n'_i, n''_i)$ is the number of ith-type sites on A(B, C). The ice-rule vertex model where there are two arrows entering each vertex corresponds to the case

$$\omega_i=\omega'_i=\omega''_i=0 \qquad \text{if } i=7,8. \tag{3}$$

† Supported by the National Science Council, Republic of China.

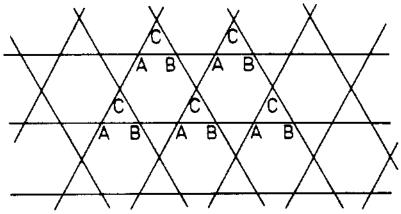

Figure 1. The Kagomé lattice with three sublattices A, B and C.

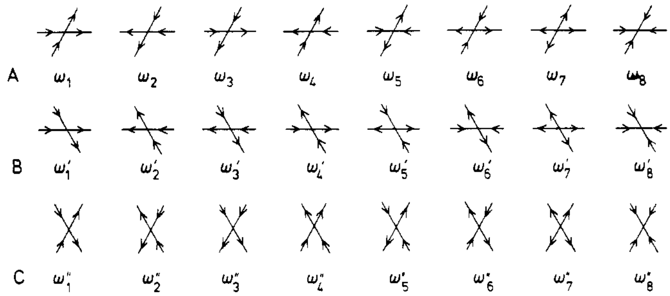

Figure 2. The 8-vertex configurations and the corresponding vertex weights on a Kagomé lattice.

The F model is a special case of the ice-rule vertex model such that the vertex configurations which differ only by rotation and reflection are treated alike. In the Kagomé lattice F model we define

$$\begin{aligned} \alpha &\equiv \omega_1 = \omega_2 = \omega_1' = \omega_2' = \omega_3'' = \omega_4'' \\ \beta &\equiv \omega_3 = \omega_4 = \omega_3' = \omega_4' = \omega_1'' = \omega_2'' \\ \gamma &\equiv \omega_5 = \omega_6 = \omega_5' = \omega_6' = \omega_5'' = \omega_6''. \end{aligned} \tag{4}$$

Place arrows on the bonds of a triangular lattice and allow only those configurations with an odd number of arrows pointing into each vertex. The 32 possible configurations allowed at each vertex and the corresponding vertex weights u_i are shown in figure 3. The ice-type 20-vertex model where there are three arrows entering and leaving each vertex corresponds to the case

$$u_i = 0 \qquad \text{if } i > 20. \tag{5}$$

The triangular lattice F model corresponds to

$$u_i = \begin{cases} \mathrm{a} & \text{if } i = 1, \dots, 6 \\ \mathrm{b} & \text{if } i = 7, 8 \\ \mathrm{c} & \text{if } i = 9, \dots, 20 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

When the vertex weights satisfy the condition

$$(a - c)^2 = a(b - c) \tag{7}$$

the F model can be solved exactly by the method of Bethe *ansatz* (Baxter 1969).

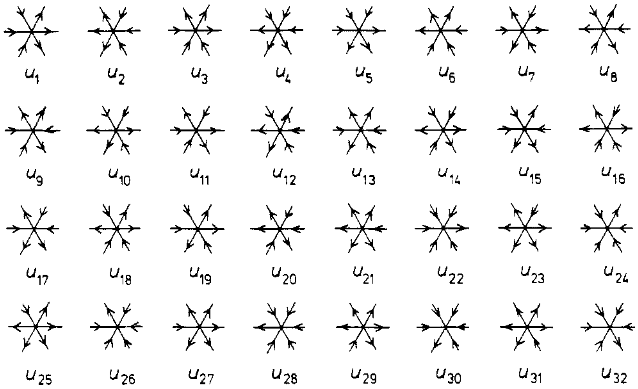

Figure 3. The 32-vertex configurations and the corresponding vertex weights on a triangular lattice.

The 8-vertex model on a Kagomé lattice with $3N$ sites is equivalent to the 32-vertex model on a triangular lattice with N sites, and the vertex weights are related by

$$\begin{array}{lll}
u_1=(111)+(875) & u_2=(222)+(786) & u_3=(144)+(857) \\
u_4=(233)+(768) & u_5=(313)+(578) & u_6=(424)+(687) \\
u_7=(342)+(556) & u_8=(431)+(665) & u_9=(164)+(837) \\
u_{10}=(253)+(748) & u_{11}=(116)+(872) & u_{12}=(225)+(781) \\
u_{13}=(387)+(524) & u_{14}=(478)+(613) & u_{15}=(365)+(531) \\
u_{16}=(456)+(642) & u_{17}=(362)+(536) & u_{18}=(451)+(645) \\
u_{19}=(345)+(551) & u_{20}=(436)+(662) & u_{21}=(237)+(764) \\
u_{22}=(148)+(853) & u_{23}=(272)+(716) & u_{24}=(181)+(825) \\
u_{25}=(275)+(711) & u_{26}=(186)+(822) & u_{27}=(317)+(574) \\
u_{28}=(428)+(683) & u_{29}=(474)+(617) & u_{30}=(383)+(528) \\
u_{31}=(257)+(744) & u_{32}=(168)+(833) &
\end{array} \qquad (8)$$

where

$$(ijk)\equiv(\omega_i\omega'_j\omega''_k).$$

A simple way to see this is to replace each allowed vertex configuration on a triangular lattice by three connected vertices on a Kagomé lattice. An example is shown in figure 4.

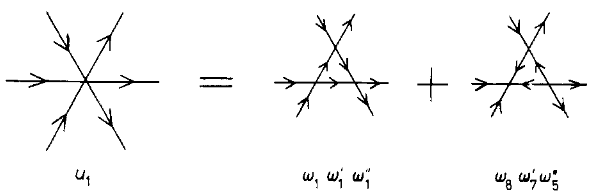

Figure 4. An example to show the equivalence of the 8-vertex model on a Kagomé lattice with the 32-vertex model on a triangular lattice.

It follows from equations (8) that $u_i = 0$ ($i > 20$) if $\omega_l = \omega_l' = \omega_l'' = 0$ ($l > 6$). Therefore the ice-rule vertex model on a Kagomé lattice is equivalent to the 20-vertex model on a triangular lattice. The F model on a Kagomé lattice is equivalent to a special case of the 20-vertex model on a triangular lattice such that

$$u_i = \begin{cases} \alpha^2\beta & \text{if } i = 1, \ldots, 6 \\ \beta^3 + \gamma^3 & \text{if } i = 7, 8 \\ \alpha^2\gamma & \text{if } i = 9, \ldots, 14 \\ \beta^2\gamma + \beta\gamma^2 & \text{if } i = 15, \ldots, 20. \end{cases} \tag{9}$$

It follows from equation (9) that the Kagomé lattice F model is equivalent to the triangular lattice F model when the vertex weights satisfy

$$\alpha^2 = \beta(\beta + \gamma) \tag{10}$$

and

$$a = \alpha^2\beta \qquad b = \beta^3 + \gamma^3 \qquad c = \alpha^2\gamma. \tag{11}$$

Note that the condition (10) for the Kagomé lattice F model implies the soluble condition (7) for the corresponding triangular lattice F model.

The ice model on a Kagomé lattice ($\alpha = \beta = \gamma = 1$) does not satisfy equation (10) and is related to the triangular ice model ($a = b = c = 1$) by

$$W(\text{Kagomé lattice}) < (2\,W(\text{triangular lattice}))^{1/3} = 3^{1/2} \tag{12}$$

where $W = \lim_{N\to\infty} Z^{1/N}$ and N is the total number of vertices.

References

Baxter R J 1969 *J. Math. Phys.* **10** 1211–6
—— 1971 *Phys. Rev. Lett.* **26** 832–3
Kelland S B 1974a *J. Phys. A: Math., Nucl. Gen.* **7** 1907–12
—— 1974b *Aust. J. Phys.* **27** 813–9
Lin K Y 1975 *J. Phys. A: Math. Gen.* **8** 1899–919
—— 1976 *J. Phys. A: Math. Gen.* **9** 581–91
Lin K Y and Tang D L 1976 *J. Phys. A: Math. Gen.* **9** 1101–7
Rys F 1963 *Helv. Phys. Acta* **36** 537–59
Sacco J E and Wu F Y 1975 *J. Phys. A: Math. Gen.* **8** 1780–7